REQUIRED MATHEMATICAL SKILLS FOR ENTERING CADETS

The Department of Applied Mathematics administers a Math Placement test to assess fundamental skills in mathematics that are necessary to begin the study of calculus. These skills are listed below in Sections 1-13 along with a sample question. We encourage incoming cadets to evaluate their mathematics skills and work at remediation of any deficiencies through completing the additional problems on the following pages and reviewing material from high school.

Note: The placement exam is to be taken on-line with a two-hour time limit without the use of technology (i.e., calculator). Instructions for completing this placement test will be provided.

1. Algebra and Real Numbers
   (a) Use symbols and operators to represent ideas and objects and the relationships existing between them.
   (b) Understand the relationship between measures of the physical world. Velocity, distance, and time: On a 40-mile car trip to Roanoke, VA, you drive the first twenty miles at 40 mph and the last twenty miles at 60 mph. What is your average speed during the trip?
   SOLUTION: First 20 miles takes 30 minutes and last 20 miles takes 20 minutes, so the total trip is 40 miles in 50 minutes, or 40*60/50 = 48mph
   (c) Know and apply the following algebraic properties of the real number system: identity, associative, commutative, inverse, and distributive.
   (d) Express numbers using scientific notation. Express 0.004312 in scientific notation.
   SOLUTION: $4.312 \cdot 10^{-3}$ or $4.312e - 3$
2. Radicals and Exponents
   (a) Convert between radical and rational exponent form. Transform $\dfrac{1}{\sqrt{x+2}}$ to the rational exponent form $(x+2)^{-1/2}$
   (b) Manipulate algebraic expressions that contain integer and rational exponents. Simplify $4^{-3/2}27^{-2/3}$
   SOLUTION: $4^{-3/2}27^{-2/3} = \dfrac{1}{8} \cdot \dfrac{1}{9} = \dfrac{1}{72}$
3. Algebraic Expressions
   (a) Add, subtract, multiply, and divide algebraic expressions. Find the remainder when $x^3 - 7x^2 + 9x$ is divided by $x - 2$.
   SOLUTION: Long division gives remainder $\dfrac{-2}{x-2}$
   (b) Simplify algebraic expressions. Expand and simplify $(x-3)(x-2)(x-1)$.
   SOLUTION: Distributive property (repeated) gives $x^3 - 6x^2 + 11x - 6$.
4. Factoring
   (a) Solve for the roots of a polynomial by factoring. Find the roots of $x^2 - 5x + 6 = 0$.
   SOLUTION: Factor as $(x-3)(x-2) = 0$ to get roots $x = 3$ and $x = 2$.
5. Linear Equations, Inequalities and Absolute Values
   (a) Solve two simultaneous linear equations by graphing and by substitution. Use a graph to estimate the point of intersection of the lines $2x + 3y = 7$ and $x + y = 4$. Verify your result using back substitution.
   SOLUTION: $x = 5$, $y = -1$.
   (b) Solve linear equations and inequalities [graphically and algebraically]. Solve $5(3-x) > 2(3-2x)$ for x.
   SOLUTION: Multiply out as $15 - 5x > 6 - 4x$ and add $5x$ to both sides to reach $9 > x$.
   (c) Solve linear equations and inequalities with absolute values. Solve $|x-4| \geq 3$ for x.
   SOLUTION:For this to happen, either $x - 4 > 3$ or $x - 4 < -3$, leading to either $x > 7$ or $x < -7$.
6. Polynomial and Rational Inequalities

(a) Solve simple polynomial inequalities. Solve $x^2 + 3x + 6 > x - 4$ for x.
SOLUTION: Bring all terms to the left for $x^2 + 2x + 10 > 0$ which is true for all x.

(b) Solve simple rational inequalities. Solve $\frac{x-3}{x+1} < 2$ for x.
SOLUTION: Subtract $\frac{x-3}{x+1}$ to the right side and get a common denominator to get $\frac{x-5}{x+1} > 0$ so either $x > 5$ or $x < -1$.

7. Lines

(a) Determine the equation of a line. Find the equation of a straight line passing through the points (2, 1) and (5, 4).
SOLUTION: Slope is 1, so the line is $y - 1 = 1(x - 2)$.

(b) Determine the equation of a line that is parallel or perpendicular to a given line. Find the equation of a line parallel to the line $2y - 3x = 7$ and passing through the point $(1, 2)$.
SOLUTION: The slope is $3/2$ so the line is $(y - 2) = (3/2)(x - 1)$.

8. Functions

   (a) Identify the independent and dependent variables of a function.
   (b) Determine the domain and range of a real valued function. Find the domain and range of the real valued function $g(x) = \frac{1}{x^2 - 2}$.
   SOLUTION: Domain is all points so that $x \neq \pm\sqrt{2}$ and the range is all real numbers.
   (c) Evaluate a function at a point. Given $f(x) = 3x^2 - 2x + 4$, find $f(2a)$.
   SOLUTION: Simply substitute $2a$ into the function and get $3(2a)^2 - 2(2a) + 4$.
   (d) Evaluate composite functions. Given $h(r) = 3r^2$ and $g(s) = 2s$, find $h(g(a + 2))$.
   SOLUTION: $g(a + 2) = 2a + 4$ and so $h(g(a + 2)) = 3(2a + 4)^2$.

9. Quadratic Equations and Systems

   (a) Solve for real and complex roots using the quadratic formula. Solve $3x^2 + 2x = -1$.
   SOLUTION: In the quadratic formula for this one, $a = 3, b = 2, c = 1$ to get

$$x = \frac{-2 \pm \sqrt{4 - 2(3)(1)}}{2(3)}$$

   (b) Solve a system of quadratic equations in 2 variables by substitution. Solve the system $y = 3 - x^2$ and $y = 4 + 2x^2 - 2x$.
   SOLUTION: Sub in y to the 2nd equation to get $3 - x^2 = 4 + 2x^2 - 2x$ or $3x^2 - 2x + 1 = 0$ so that the solution is similar to the last one:

$$x = \frac{2 \pm \sqrt{4 - 2(3)(1)}}{2(3)}$$

10. Logarithmic and Exponential Functions

   (a) Know the relationship between logarithm and exponential functions [$y = \log_a x, a > 0, a \neq 1$ is the inverse of the function $y = a^x, \log ax = y \leftrightarrow a^y = x$] Evaluate $\log_3 27$.
   SOLUTION: Answer is 3 since $3^3 = 27$.
   (b) Know the properties of the logarithmic and exponential functions and use them to simplify logarithmic expressions. Express as a single logarithm: $0.5 \log_{10} x - \log_{10} y$.
   SOLUTION: $\log_{10}\left(\frac{\sqrt{x}}{y}\right)$.
   (c) Solve simple logarithmic and exponential equations. Solve the equation $3^{x+4} = 4$ for x.
   SOLUTION: Take the logarithm base 3 of both sides and subtract 4 to get $x = (\log_3 4) - 4$

11. Trigonometric Functions

   (a) Define each of the 6 trigonometric functions $\sin(\theta), \cos(\theta), \tan(\theta), \cot(\theta), \sec(\theta), \csc(\theta)$ in terms of the sides of a right triangle. For example, $\cos(\theta) = \frac{A}{H}$ where A is the adjacent side and H is the hypotenuse.
   SOLUTION: $\sin(\theta) = \frac{O}{H}, \cos(\theta) = \frac{A}{H}, \tan(\theta) = \frac{O}{A}$ and $\csc(\theta), \sec(\theta), \cot(\theta)$ are the reciprocals of these, resp.
   (b) Define each of the 6 trigonometric functions in terms of $\sin(\theta)$ and $\cos(\theta)$. For instance, $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$
   SOLUTION: $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}, \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}, \sec(\theta) = \frac{1}{\cos(\theta)}, \csc(\theta) = \frac{1}{\sin(\theta)}$
   (c) Know the domain and ranges for the sine, cosine, and tangent functions.
   SOLUTION: The domain for sine and cosine are all real numbers and the range is the interval $-1 \leq y \leq 1$. The domain for tangent is all values where cosine is not zero ($\pm$ odd multiples of $\pi/2$ where there are asymptotes) and the range is all real values.
   (d) Convert angle measures between degrees and radians. (Write 120 degrees as a radian measure.)
   SOLUTION: 1 radian is $180/\pi$ degrees and 1 degree is $\pi/180$ radians. 120 degrees is $2\pi/3$ radians.
   (e) Memorize and use the 30/60/90 and 45/45/90 degree reference triangles.

(f) Know and apply the trigonometric identity $\sin^2(\theta) + \cos^2(\theta) = 1$. Simplify the expression $2\cos^2(\theta) + \sin^2(\theta) - 1$.
SOLUTION: $2\cos^2(\theta)+\sin^2(\theta)-1 = \cos^2(\theta)+(\cos^2(\theta)+\sin^2(\theta))-1 = \cos^2(\theta)+(1)-1 = \cos^2(\theta)$

12. Graphs and Graphing

(a) Graph equations and inequalities. Sketch a graph of the function $f(x) = 3x^2-2x+7$ for $1 < x < 5$.
SOLUTION: This will be part of a parabola.

(b) Properly label a graph (axes, intercepts, asymptotes, and roots).

(c) Know the general characteristics and shapes of the graphs of polynomial, logarithm, exponential and trigonometric functions.

(d) Transform the graph of a known function. From the graph of $f(x)$, graph the function $g(x) = 2f(x) - 3$.
SOLUTION: This doubles the y values and then shifts the graph down 3 units.

13. Analytic Geometry

(a) Know and apply the distance formula between 2 points. Find the distance between the two points $A(1, 2)$ and $B(5, 3)$.
SOLUTION: $\sqrt{(-3-2)^2 + (5-1)^2} = \sqrt{25+16} = \sqrt{41}$.

(b) Know and apply the circumference and area formulas for circles and the perimeter and area formulas for triangles and rectangles. If you double the radius of a circle, what happens to its circumference?
SOLUTION: Doubling the radius will double the circumference since $C = 2\pi r$.

(c) Know and apply the surface area and volume formulas for cylinders, spheres and rectangular solids.

(d) Know the relationship between similar triangles. A rectangle with base x and height 5 is inscribed in an isosceles triangle with base 10 and height 20. Determine x.
SOLUTION: The similar triangles would give 5 is to 20 as x is to 5, so $x = 5/4$.

(e) Know and apply the Pythagorean Theorem to simple geometric problems. Given a rectangle that is 4 ft by 7 ft, determine the length of the diagonal.
SOLUTION: The diagonal, using the Pythagorean Theorem, is $\sqrt{4^2 + 7^2} = \sqrt{65}$.

Algebra and Exponents

Some Arithmetic Properties of the Real Numbers

$a + b = b + a$ (commutativity of addition)
$ab = ba$ (commutativity of multiplication)
$a(b + c) = ab + ac$ (distributivity)
$(a + b)c = ac + bc$ (distributivity)
$a^{-1} = \frac{1}{a}$ (division)
$b^a b^c = b^{a+c}$ (law of exponents)

Simplify the following expressions as much as possible, using only positive exponents in your final answer:

1. $\frac{4-18}{14-7}$
   SOLUTION: $\frac{-14}{7} = -2$

2. $\frac{2^{13}\,3^{1/2}}{3^{-1/2}\,2^{10}}$
   SOLUTION: $2^3 \cdot 3^1 = 24$

3. $4^{1/2}\sqrt{2^2}$
   SOLUTION:

4. $\frac{1/2}{1/4}$
   SOLUTION: $2 \cdot 2 = 4$

5. $\frac{\frac{2x+1}{13}}{\frac{4x+2}{5}}$
   SOLUTION: Flip and multiply: $\frac{2x+1}{13}\frac{5}{4x+2} = \frac{5}{13 \cdot 2} = \frac{5}{26}$

6. $(-4)^3$
   SOLUTION: -64

7. $(2 + 3 + 4)(2 - 3 - 4)$
   SOLUTION: $9 \cdot -7 = -63$

8. $\frac{5^2 - 4^2}{2}$
   SOLUTION: $9/2$

9. $\frac{(-1)^3(2-12)}{10^2}$
   SOLUTION: $\frac{10}{10^2} = \frac{1}{10}$

10. $\frac{2^2 - 3^2}{3^3 - 2^0}$
    SOLUTION: $\frac{-5}{26}$

Simplify the following expressions as much as possible:

1. $-4a(a + b - c)$
   SOLUTION: $-4a^2 - 4ab + 4ac$

2. $\frac{a+b}{(a+b)^{-1}}$
   SOLUTION: $(a + b)^2$ or $a^2 + 2ab + b^2$

3. $\frac{a^2 - b^2}{a - b}$, where $a \neq b$
   SOLUTION: $a + b$

4. $(2+a+c)(a-c)$
   SOLUTION: $a^2+2a-2c-c^2$

5. $\frac{\frac{a^2+a}{b}}{\frac{b^2}{a}}$
   SOLUTION: $\dfrac{a^3+a^2}{b^3}$

Factoring

Factoring Quadratics and the Quadratic Equation

The general form for a quadratic equation is $ax^2 + bx + c = 0$ where $a \neq 0$. If the coefficients a, b, and c are integers, we may look for solutions to this equation by factoring. This means we find integers d, e, f, g and an expression $(dx+e)(fx+g)$ such that $df = a, eg = c$, and $dg + ef = b$.

If the coefficients a, b, and c are real numbers, we may find solutions to the quadratic equation by using the quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

NOTE: We are not always able to factor quadratics, and sometimes solutions given by the quadratic formula are complex numbers.
Determine if the following statements are true or false, (you must justify your answer mathematically):

1. $a^2 + ac + c^2 = (a+c)^2$
   SOLUTION: False. The middle term should be $2ac$.
2. $a^2 - ac + c^2 = (a-c)^2$
   SOLUTION: False. The middle term should be $-2ac$.
3. $(a+c)(b+d) = ab + 2ad + cd$
   SOLUTION: False, the right side should be $ab + ad + bc + cd$.
4. $(a-c)(a+b) = a^2 + ab - ac - cb$
   SOLUTION: True
5. $a^2 - c^2 = (a-c)^2$
   SOLUTION: False, $(a-c)^2 = a^2 - 2ac + c^2$.

If possible, factor the following polynomials. Otherwise find solutions to the associated quadratic equation using the quadratic formula.

1. $x^2 + 7x + 12$
   SOLUTION: $(x+3)(x+4)$
2. $x^2 - 8x - 9$
   SOLUTION: $(x-9)(x+1)$
3. $x^2 + 5x + 6$
   SOLUTION: $(x+3)(x+2)$
4. $x^2 - 36$
   SOLUTION: $(x+6)(x-6)$
5. $x^2 + 7x - 6$
   SOLUTION: Using the quadratic theorem, the roots are $\dfrac{-7 \pm \sqrt{49+24}}{2}$ and if you call these roots a and b then it will factor as $(x-a)(x-b)$.
6. $2x^2 + x - 6$
   SOLUTION: $(x+2)(2x-3)$
7. $2x^3 + x^2 - 6x$
   SOLUTION: $x(x+2)(2x-3)$
8. $x^2 - 19x + 12$
   SOLUTION: Using the quadratic theorem, the roots are $\dfrac{19 \pm \sqrt{19^2 - 48}}{2}$ and if you call these roots a and b then it will factor as $(x-a)(x-b)$.
9. $5x^2 - 10x + 6$
   SOLUTION: This does not factor.
10. $x^2 - 2x - 8$
    SOLUTION: $(x+2)(x-4)$

11. $x^2 + 36$
SOLUTION: This does not factor.

12. $6x^2 - x + 2$
SOLUTION: This does not factor

13. $x^2 - 2$
SOLUTION: $(x + \sqrt{2})(x - \sqrt{2})$

14. $6 + x - x^2$
SOLUTION: $(x + 2)(-x + 3)$

15. $64 - 4x^2$
SOLUTION: $(8 - 2x)(8 + 2x)$

Linear Inequalities

Inequalities and Intervals

Suppose $a < b$.
Fact A: If $c > 0$, then $ca < cb$.
Fact B: If $c < 0$, then $ca > cb$.
Fact C: If a and b are both positive or both negative, then $\frac{1}{a} > \frac{1}{b}$
Fact D: If $a < 0 < b$, then $\frac{1}{a} < \frac{1}{b}$
Fact E: The open interval (a, b) denotes the set $\{a < x < b.\}$
Fact F: The closed interval $[a, b]$ denotes the set $\{a \leq x \leq b\}$.
Fact G: The half-open interval $[a, b)$ denotes the set $\{a \leq x < b\}$.
Fact H: The half-open interval $(a, b]$ denotes the set $\{a < x \leq b\}$.

Determine if the following statements are true or false. You must justify your answer mathematically.

1. If $a < b$ and $b < c$, then $a < c$.
   SOLUTION: True, this is the transitive property.

2. If $a < b$ then $a + c < b + c$.
   SOLUTION: True.

3. If $a < b$ and $c < d$, then $a + c < b + d$.
   SOLUTION: True

4. If $a < b$ and c is a real number, then $ac < bc$.
   SOLUTION: This is true if $c > 0$ but false if $c \leq 0$.

5. If $1 < x^2$, then $1 < x$.
   SOLUTION: False, since $x < -1$ is possible.

Solve the following problems:

1. If $x < 2$ and $x \geq -1$, write the corresponding interval.
   SOLUTION: $[-1, 2)$

2. If ≤ 10 and $x \geq -11$, write the corresponding interval.
   SOLUTION: $[-11, 10]$

3. If $2x \leq 10$ and $x + 5 \geq -11$, write the corresponding interval.
   SOLUTION: $[-16, 5]$

4. If $3x < 5$ and $3x > -7$, write the corresponding interval.
   SOLUTION: $(-7/3, 5/3)$

5. If $ax \leq b$ and $dx > c$ with $a, d > 0$, write the corresponding interval.
   SOLUTION: $(c/d, b/a]$

Solve the following problems, using the symbols $+\infty$ and $-\infty$ as appropriate:

1. Write an interval accurately expressing the set: $\{x : x > a\}$.
   SOLUTION: (a, ∞)

2. Write an interval accurately expressing the set: $\{x : x < a\}$.
   SOLUTION: $(-\infty, a)$

3. Write an interval accurately expressing the set: $\{x : x \geq a\}$.
   SOLUTION: $[a, \infty)$

4. Write an interval accurately expressing the set: $\{x : x \leq a\}$.
   SOLUTION: $(-\infty, a]$

5. Write an interval accurately expressing the set: $\{x : a \leq x \leq a\}$.
   SOLUTION: $[a, a]$ or just the point $\{a\}$.

Solve the following problems:

1. If $x + 3 < 2$, write the corresponding interval or intervals.
   SOLUTION: $(-\infty, -1)$

2. If $4x + 5 < 2x - 7$, write the corresponding interval or intervals.
   SOLUTION: $(-\infty, -13/2)$

3. If $ax - b \leq cx + d$, write the corresponding interval or intervals.
   SOLUTION: There are three possibilities for a and c:
   If $a = c$, then any solution works if $-b = d$ and no solution if $-b \neq d$
   If $a > c$, then the solution is $\left(-\infty, \dfrac{b+d}{a-c}\right]$
   If $c > a$, then the solution is $\left[\dfrac{-b-d}{c-a}, \infty\right)$

4. If $x^2 \geq 1$, write the corresponding interval or intervals.
   SOLUTION: $(-\infty, -1] \cup [1, \infty)$.

Absolute Values

The Absolute Value

The absolute value of x is defined as

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

For any constants, variables, or functions a and b, we have the following property of the absolute value function: $|ab| = |a||b|$.
For a real number $a, a > 0$, from the comments above, we know the following statements are equivalent: $|x| \leq a \leftrightarrow -a \leq x \leq a$.
Let a, b, c be real numbers, where $a > 0$ and $c \neq 0$. Determine if the following statements are true or false. You must justify your answer mathematically.

1. If $|x| < 2$, then $x < 2$ and $x > -2$.
   SOLUTION: False, it should be $\sqrt{2} < x < \sqrt{2}$.

2. If $|x| < a^2$, then $-a < x < a$.
   SOLUTION: True

3. If $x < 3$ and $x \geq 2$, then $|x| \leq 2$.
   SOLUTION: False

4. If $a < b$, then $|c|a < |c|b$.
   SOLUTION: True

5. If $|x - 2| \leq 4$, then $|x| \leq 6$.
   SOLUTION: True

Let a, b, c be real numbers, where $a > 0$ and $c \neq 0$. Write the following sets as intervals:

1. $\{x : |x - 5| < 1\}$
   SOLUTION: $4 < x < 6$.

2. $\{x : |x| < a\}$
   SOLUTION: $-a < x < a$

3. $\{x : |x - c| \leq a\}$
   SOLUTION: $-a + c \leq x \leq a + c$

4. $\{x : |2x - 4| \leq 7\}$
   SOLUTION: $-3/2 \leq x \leq 11/2$

5. $\{x : |3x - 2| < \frac{1}{4}\}$
   SOLUTION: $7/6 < x < 3/4$

Let a, b, c be real numbers, where $a > 0$ and $c \neq 0$. Determine if the following statements are true or false. You must justify your answer mathematically.

1. $|-b| = |b|$
   SOLUTION: True

2. $|a|\left|\frac{1}{c}\right| = \frac{|a|}{|c|}$
   SOLUTION: True

3. $|(-3)^3| = -27$.
   SOLUTION: False, it should be 27

4. The function $|5x - 10|$ may be written as:

$$|5x - 10| = \begin{cases} 5x - 10, & 5x - 10 \geq 0, \\ -(5x - 10), & 5x - 10 < 0 \end{cases}$$

   SOLUTION: True

5. The function $|5x - 10|$ may be written as:

$$|5x - 10| = \begin{cases} 5x - 10, & x \geq 2 \\ -5x + 10, & x < 2 \end{cases}$$

SOLUTION: True

6. $|-x||-x| = |x|$
SOLUTION: False, the left side is x^2

7. $\dfrac{|a|}{|c|} = |a||c|^{-1}$
SOLUTION: True as long as $c \neq 0$

8. If $|3x - 5| = 10$, then $x = 5$ or $x = -\dfrac{5}{3}$.
SOLUTION: True

9. If $\dfrac{|a|}{|c|} = |b|$, then $|a| = b|c|$.
SOLUTION: False if $b < 0$

10. If $\dfrac{|x+1|}{|x-1|} = 2$, then $x = \frac{1}{3}$ is the only solution.
SOLUTION: False, $x = 3$ also works.

Polynomial

Polynomials

A polynomial (with real coefficients) is a function $p(x)$ of the form: $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ where x is a variable, all a_i are real numbers, and $a_n \neq 0$. We say the degree of this polynomial is n, that is, the maximum power of x in $p(x)$. A number r is called a root or zero of the polynomial $p(x)$ if $p(r) = 0$.

Determine if the following statements are true or false. You must justify your answer mathematically.

1. $f(x) = x^3 + 5x^2 + 1$ has degree 3.
   SOLUTION: True

2. $g(x) = 13x^{12} + 12x^{13} + 11$ has degree 12.
   SOLUTION: True

3. The graph of a degree 1 polynomial is a straight line.
   SOLUTION: True

4. The graph of a degree 0 polynomial is a vertical line.
   SOLUTION: False

5. The graph of a degree 0 polynomial is a horizontal line.
   SOLUTION: True

Solve the following problems:

1. Give one word to describe the graph of a degree 2 polynomial.
   SOLUTION: Parabola

2. Answer true or false: $f(x) = 13.2$ is a polynomial.
   SOLUTION: Treu (degree zero)

3. Answer true or false: $f(x) = mx + b$ is a polynomial, where b, m are real numbers.
   SOLUTION: True, a monomial

4. What is the degree of the polynomial $p(x) + q(x)$, where $p(x) = x^3 + 2x + 13$ and $q(x) = 17x^2$?
   SOLUTION: Three

5. What is the degree of the polynomial $p(x)q(x)$, where $p(x) = x^3$ and $q(x) = 17x^2$?
   SOLUTION: Five

Solve the following problems:

1. What is the degree of the polynomial $p(x)q(x)$, where $p(x) = x^4 + 1$ and $q(x) = x^4 + 7$?
   SOLUTION:

2. What is the degree of the polynomial $p(x)q(x)$, where $p(x) = 11x^7 + x^6 + 9x^3 + 1$ and $q(x) = 22x^{13} + 5x^3 + 2x + 7$?
   SOLUTION: Eight

3. Answer true or false: if $f(x)$ is quadratic, then f(x) is guaranteed to have two distinct real roots and we have a method to find the real values of those roots.
   SOLUTION: False, there could be one or no roots, too.

4. Answer true or false: if $f(x)$ is linear, how many roots may $f(x)$ have?
   SOLUTION: Exactly one.

5. Answer true or false: if $f(x)$ has degree 3, then $f(x)$ must cross the x-axis.
   SOLUTION: True

Find the real roots of the following polynomials.

1. $f(x) = (x + 2)(x - 1)$
   SOLUTION: $x = 1$ and $x = -2$

2. $g(x) = x^2 - 7$
   SOLUTION: $x = \pm\sqrt{7}$

3. $h(x) = 1$
   SOLUTION: None
4. $p(x) = x^3 - 4x^2 - 7x + 10$
   SOLUTION: $x = 1$, $x = 5$ and $x = -2$
5. $q(x) = x^4 - 1$
   SOLUTION: $x = 1$

Lines

Lines

Two of the most useful forms for a straight line are the Point-Slope Form and the Slope-Intercept Form.

Given points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, the line passing through both P_1 and P_2 may be found using the Point-Slope Form: $y - y_1 = m(x - x_1)$ where $m = \frac{y_2 - y_1}{x_2 - x_1}$.

If m is the slope of a line, and b is it's y-intercept, the Slope-Intercept Form of that line is given by: $y = mx + b$

Determine if the following statements are true or false, (you must justify your answer mathematically):

1. The line $x = 0$ is horizontal.
   SOLUTION: False, it is vertical.
2. The line $x - 3 = 4$ is vertical.
   SOLUTION: Treu
3. The line $y = \pi$ is horizontal.
   SOLUTION: Treu
4. The line $2y - y = 3 + y + x$ is horizontal.
   SOLUTION: False
5. Two distinct lines must always intersect at a single point.
   SOLUTION: No, they could be parallel or the same line.

Solve the following problems:

1. Given the general point-slope form of a line above, find b for that line in its slope-intercept form.
   SOLUTION: Multiply out to get it into the form of the $y = mx + b$
2. Find the slope of the line through the points $P_1(1, 1)$ and $P_2(2, 3)$.
   SOLUTION: $m = 2$
3. Find the slope of the line through the points $P_1(-1, -1)$ and $P_2(0, 0)$.
   SOLUTION: $m = 1$
4. Find the slope of the line through the points $P_1(6, -7)$ and $P_2(4, -7)$.
   SOLUTION: $m = 0$
5. Find the slope of the line through the points $P_1(7, 5)$ and $P_2(4, -8)$.
   SOLUTION: $m = 13/3$

Find the equation for the line through the given points. Put the line into slope-intercept form.

1. $P_1(1, 1)$ and $P_2(2, 3)$
   SOLUTION: $(y - 1) = 2(x - 1)$
2. $P_1(-1, -1)$ and $P_2(0, 0)$
   SOLUTION: $(y - 0) = 1(x - 0)$
3. $P_1(6, -7)$ and $P_2(4, -7)$
   SOLUTION: $(y + 7) = 0(x - 6)$
4. $P_1(7, 5)$ and $P_2(4, -8)$
   SOLUTION: $(y - 7) = 13/3(x - 5)$
5. $P_1(a, b)$ and $P_2(a, d)$, where $a \neq c$.
   SOLUTION: $x = a$

For the following pairs of lines, determine if they intersect. If they intersect, state precisely where.

1. $y = x + 1, y = -x + 1$
   SOLUTION: $x = 0$

2. $2x - 3y = 6, 3x + 2y = 48$
   SOLUTION: $x = 160/13$

3. $12x - 15y = 60, 8x - 10y = 40$
   SOLUTION: They are the same line

4. $5x + 2y = 12, 4x - 7 = y$
   SOLUTION: $x = 2$

5. $y = 2x + 3, 2x - y = 10$
   SOLUTION: They don't intersect.

Functions

Functions and Composition of Functions

A function is a rule that assigns to each element of a set D (the domain) exactly one element from a set R (the range). Intuitively, when composing functions we may think of the action as "putting one function inside another function". More accurately, we say that we evaluate the function $f(x)$ at $g(x)$ when we compose f with g, and we write $f(g(x))$ or $(f \circ g)(x)$ to denote the composition.

A function f has a (functional) inverse g if $g(f(x)) = x$ for every x in the domain of f. We usually denote this inverse function by f^{-1} rather than g. For example, e^x and $\ln(x)$ are inverse functions. So, $e^{\ln(x)} = x$ and $\ln(e^x) = x$, for all x in the domains of $\ln(x)$ and e^x respectively.

For all problems below, assume $f(x) = x^2, g(x) = e^x$, and $h(x) = 3$.

Determine if the following statements are true or false, (you must justify your answer mathematically):

1. $f(g(x)) = e^{x^2}$
   SOLUTION: False, $f(g(x)) = (e^x)^2$
2. $f(g(x)) = e^{2x}$
   SOLUTION: True
3. $f(h(x)) = 3$
   SOLUTION: False, the answer is 9
4. $h(f(x)) = 3$
   SOLUTION: True
5. $\ln(g(x)) = x$
   SOLUTION: True

Solve the following problems, and simplify your answers:

1. $h(h(4))$
   SOLUTION: 3
2. $f(4) \cdot g(4)$
   SOLUTION: $16e^4$
3. $h(4)h(4)$
   SOLUTION: 9
4. $g(f(x))$
   SOLUTION: e^{x^2}
5. $f(g(h(1)))$
   SOLUTION: e^6

Let $k(x) = x^3 + 7x + 1$. Solve the following problems, and simplify your answers:

1. $f(1) \cdot g(1)$
   SOLUTION: e
2. $k(h(13))$
   SOLUTION: $k(3) = 49$
3. $k(g(0))$
   SOLUTION: $k(1) = 9$
4. $f(k(1))$
   SOLUTION: 81
5. $g(k(1))k(g(0))$
   SOLUTION: $9e^9$

Let $k(x) = x^3 + 7x + 1$, and a be a real number. Solve the following problems, and simplify your answers:

1. $\dfrac{k(1)f(1)}{f(1)g(1)}$
   SOLUTION: $9/e$

2. $\dfrac{k(f(1))}{g(f(1))}$
   SOLUTION: $9e$

3. $f(2+a)$
   SOLUTION: $(2+a)^2$

4. $f(x+a)$
   SOLUTION: $(x+a)^2$

5. $\dfrac{f(x+a)-f(x)}{(x+a)-x}$
   SOLUTION: $\dfrac{(x+a)^2-x^2}{a} = 2x+a$

Logarithmic and Exponential Functions

Logarithmic and Exponential Rules

$b^a b^c = b^{a+c}, (b^a)^c = b^{ac}, a^{-1} = \frac{1}{a}, b^{\log_b a} = a, a^0 = 1 (a \neq 0)$

$\log_b(ac) = \log_b(a) + \log_b(c), \log_b(a/c) = \log_b(a) - \log_b(c), \log_b(a^c) = c\log_b(a), \log_b b^a = a, \log_b(c) = \frac{\log_a(c)}{\log_a(b)}$

Determine if the following statements are true or false. You must justify your answer mathematically.

1. $\log_2(2^2) = 1$
   SOLUTION: False, it is 2
2. $10^2 5^2 = 50^2$
   SOLUTION: True
3. $(a+b)^2 = a^2 + b^2$
   SOLUTION: False, the right side should have $2ab$
4. $(ab)^{2c} = a^2 b^c$
   SOLUTION: False, both a and b both have power $2c$
5. $\log_{10}(10^{13} 10^b) = 13^b$
   SOLUTION: False it is $13 + b$

Simplify the following expressions as much as possible:

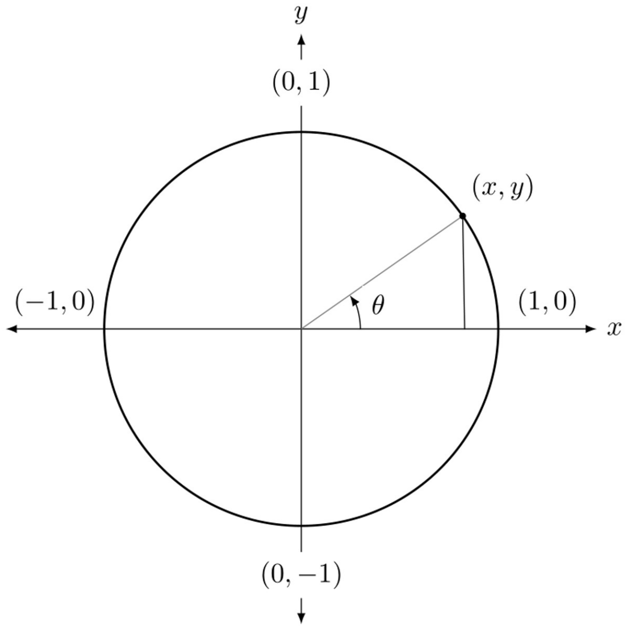
1. $\dfrac{b^4}{2^4}$
   SOLUTION: It is in lowest form although $\left(\dfrac{b}{2}\right)^4$ is another form
2. $\dfrac{40^3}{5 \cdot 8^3}$
   SOLUTION: 25
3. $\dfrac{13^2}{11^2}$
   SOLUTION: In lowest form
4. $\dfrac{1/2}{1/2^5}$
   SOLUTION: 16
5. $\dfrac{7^0 a^2 b^4}{b^{-1} a^3 7^2}$
   SOLUTION: $\dfrac{b^5}{49a}$
6. $\log_5(5^3)$
   SOLUTION: 3
7. $\log_5(25^3)$
   SOLUTION: 6
8. $\log_{13}(169)$
   SOLUTION: 2
9. $\log_{10}(1000^\pi)$
   SOLUTION: π
10. $\dfrac{\log_3(8)}{\log_3(2)}$
    SOLUTION: This is in lowest form
11. $\dfrac{\log_8(64)}{\log_5(125)}$
    SOLUTION: 2/3

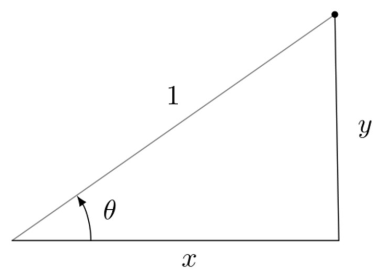
12. $\log_b(b^{10000}) - \log_b(b^{100})$
SOLUTION: $10,000 - 100 = 9,900$

13. $\log_2(25)(\log_2(8))^{-1}$
SOLUTION: $\log_2(25)/3$ is lowest form

Trigonometric Functions

Recall the unit circle





FACT A: The cartesian equation for the unit circle is: $x^2 + y^2 = 1$.
FACT B: Positive angle measurements for θ are made counter-clockwise from the positive side of the x-axis.
FACT C: A circular arc on the unit circle with angle of θ degrees has length $\theta \cdot \frac{\pi}{180}$.
FACT D: π radians equals $180°$

1. Find the coordinates for the points on the unit circle where $x = y$.
   SOLUTION: $\pm(\sqrt{2}, \sqrt{2})$
2. Find the coordinates for the points on the unit circle where $x = 2/3$
   SOLUTION: $(2/3, \pm\sqrt{5}/3)$
3. Find the coordinates for the points on the unit circle where $y = 0$.
   SOLUTION: $(\pm 1, 0)$
4. Find the coordinates for the points on the unit circle where $y = 2/5$
   SOLUTION: $(\pm\sqrt{21}/5, 2/5)$
5. Find the coordinates for the points on the unit circle that intersect a line through the origin with slope 1.
   SOLUTION: $\pm(\sqrt{2}, \sqrt{2})$

Find the actual length (in terms of a fraction times π) on the unit circle of the angles $\theta = 0°, 45°, 75°, 90°, 180°$

SOLUTION: $\theta = 0, \pi/4, 5\pi/12, \pi/2, \pi$, respectively

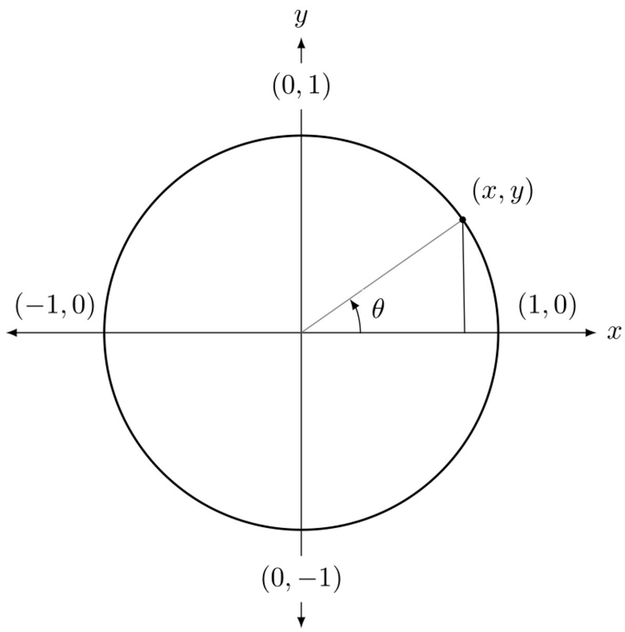
Find the angles (in degrees) for a circular arc on the unit circles with lengths $\frac{\pi}{5}, \frac{\pi}{4}, \frac{2\pi}{3}, \frac{5\pi}{6}$.

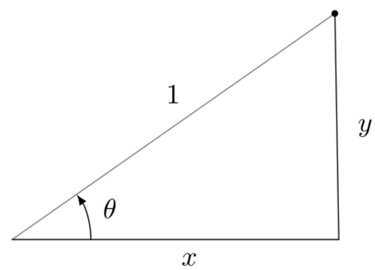
SOLUTION: $36°, 45°, 120°, 15°$ respectively.

Convert these angles to radians $45°, 15°, -60°, 270°, -360°$.

SOLUTION: $\frac{\pi}{5}, \frac{\pi}{12}, -\frac{\pi}{3}, 3\frac{\pi}{2}, -2\pi$ respectively.

Now in terms of the trigonometric functions:





FACT A: A polar equation for the unit circle is: $\sin^2\theta + cos^2\theta = 1$.
FACT B: The sine function is periodic with period 2π, so $\sin(\theta = \sin(\theta + 2n\pi)$ for any integer n.
FACT C: The cosine function is periodic with period 2π, so $\cos(\theta) = \cos(\theta + 2n\pi)$ for any integer n.
FACT D: We have the following table

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1

Fill in the table:

θ	π	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$
$\sin\theta$	0					
$\cos\theta$	-1					

SOLUTION:

θ	π	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$
$\sin\theta$	0	$\sqrt{3}/2$	$\sqrt{2}/2$	1/2	-1/2	$-\sqrt{2}/2$
$\cos\theta$	-1	-1/2	$-\sqrt{2}/2$	$-\sqrt{3}/2$	$-\sqrt{3}/2$	$-\sqrt{2}/2$

For each of the following angles, find $\sin\theta$ and $\cos\theta$.

1. $\theta = \pi/4$
   SOLUTION: $\sin\theta = -\sqrt{2}/2$ and $\cos\theta = \sqrt{2}/2$

2. $\theta = \pi/3$
   SOLUTION: $\sin\theta = -\sqrt{3}/2$ and $\cos\theta = 1/2$

3. $\theta = 3\pi/2$
   SOLUTION: $\sin\theta = -1$ and $\cos\theta = 0$

4. $\theta = 120°$
   SOLUTION: $\sin\theta = \sqrt{3}/2$ and $\cos\theta = -1/2$

5. $\theta = 300°$
   SOLUTION: $\sin\theta = \sqrt{3}/2$ and $\cos\theta = 1/2$

Answer the following questions:

1. Suppose that $\sin\theta = 3/5$ and $\pi/2 < \theta < \pi$. What is $\cos\theta$?
   SOLUTION: -4/5

2. Suppose that $\sin\theta = 3/5$ and $0 < \theta < \pi/2$. What is $\cos\theta$?
   SOLUTION: 4/5

3. Suppose that $\cos\theta = 2/5$ and $0 < \theta < \pi/2$ What is $\sin\theta$?
   SOLUTION: $\sqrt{21}/5$

4. Suppose that $\cos\theta = 2/5$ and $-\pi/2 < \theta < 0$ What is $\sin\theta$?
   SOLUTION: $-\sqrt{21}/5$

5. Suppose that $\cos\theta = 1/10$ and $-\pi/2 < \theta < 0$ What is $\sin\theta$?
   SOLUTION: $-\sqrt{99}/10$

If $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, find $\tan\theta$ for each of the following angles

1. $45°$
   SOLUTION: 1

2. 0
   SOLUTION: 0

3. $\pi/3$
   SOLUTION: $\sqrt{3}$

4. $\pi/6$
   SOLUTION: $1/\sqrt{3}$

5. $-120°$
   SOLUTION: $\sqrt{3}$

Some basic trigonometric identities

Identity Set A: $\sin^2\theta + \cos^2\theta = 1$ implies $\tan^2\theta + 1 = \sec^2\theta$, when $\cos\theta \neq 0$.
Identity Set B: The basic symmetric identities are: $\sin(-\theta) = -\sin\theta, \cos(-\theta) = \cos\theta, \tan(-\theta) = -\tan\theta$.
Identity Set C: The double-angle formulae are: $\sin(2\theta) = 2\cos\theta\sin\theta, \cos(2\theta) = \cos^2\theta - \sin^2\theta, \tan(2\theta) = \dfrac{2\tan\theta}{1-\tan^2\theta}$.

Determine if the following statements are true or false. You must justify your answer mathematically.

1. $\sin^2(\theta/2) = 1 - \cos^2(\theta/2)$.
   SOLUTION: True

2. $\cos(\theta) = \cos^2(\theta/2) - \sin^2(\theta/2)$.
   SOLUTION: False

3. $\cos(\theta) = 2\cos^2(\theta/2) - 1$.
   SOLUTION: True

4. $\cos^2(\theta/2) = \dfrac{\cos\theta + 1}{2}$
   SOLUTION: True

5. $\tan\theta = \dfrac{2\sin\theta}{2\cos\theta}$.
   SOLUTION: True

Determine if the following statements are true or false. You must justify your answer mathematically

1. The half-angle formula for cosine is: $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}$.

   SOLUTION: True

2. The half-angle formula for sine is: $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$.

   SOLUTION: True

3. $\tan\theta = \frac{2\sin\theta\cos\theta}{2\cos^2\theta}$
   SOLUTION: True

4. $\tan\theta = \frac{\sin(2\theta)}{1+\cos(2\theta)}$.
   SOLUTION: False

5. A half-angle formula for tangent is: $\tan\left(\frac{\theta}{2}\right) = \frac{\sin\theta}{1+\cos\theta}$.

   SOLUTION: True

Suppose $\sin a = \frac{\sqrt{3}}{2}$ and $\cos a = \frac{1}{2}$, then

1. Determine $\cos(2a)$.
   SOLUTION: $-1/2$

2. Determine $\sin(2a)$.
   SOLUTION: $\sqrt{3}/2$

3. Determine $\tan(a)$.
   SOLUTION: $\sqrt{3}$

4. Determine $\sin(-a)$.
   SOLUTION: $-\frac{\sqrt{3}}{2}$

5. Determine $\tan(-a)$.
   SOLUTION: $-\sqrt{3}$

Solve each of the following problems:

1. Determine $\sin\frac{\pi}{8}$.

   SOLUTION: $\frac{\sqrt{2-\sqrt{2}}}{2}$

2. Determine $\cos\frac{\pi}{8}$.

   SOLUTION: $\frac{\sqrt{2+\sqrt{2}}}{2}$

3. Determine $\tan\frac{\pi}{8}$.

   SOLUTION: $\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}}$

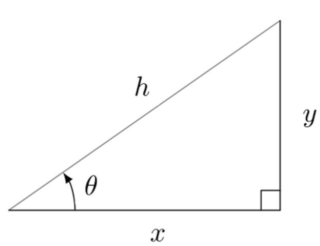
4. Determine $\cos\frac{\pi}{12}$

   SOLUTION: $\frac{\sqrt{6}+\sqrt{2}}{4}$

5. Determine $\sec^2\frac{\pi}{8}$

   SOLUTION: $\frac{4}{\sqrt{2+\sqrt{2}}}$

Now in general, we define the the trigonometric functions as ratios of sides of a triangle



FACT A: In terms of the triangle above, the Pythagorean Theorem implies: $x^2 + y^2 = h^2$.
FACT B: In terms of the triangle above, the six basic trigonometric functions are defined as:
$\sin\theta = \dfrac{y}{h}, \cos\theta = \dfrac{x}{h}, \tan\theta = \dfrac{y}{x} = \dfrac{\sin\theta}{\cos\theta}$
$\csc\theta = \dfrac{h}{y} = \dfrac{1}{\sin\theta}, \sec\theta = \dfrac{h}{x} = \dfrac{1}{\cos\theta}, \cot\theta = \dfrac{x}{y} = \dfrac{1}{\tan\theta}.$

FACT C: The sine function is periodic with period 2π, hence $\sin(\theta) = \sin(\theta + 2n\pi)$ for any integer n.
FACT D: The tangent function is periodic with period π, hence $\tan(\theta) = \tan(\theta + n\pi)$ for any integer n and any angle θ where $\cos\theta \neq 0$.
Solve each of the following problems:

1. If $y = 3$ and $h = 5$, determine x.
   SOLUTION: 4
2. If $y = 3$ and $h = 5$, determine $\sec\theta$.
   SOLUTION: 5/4
3. If $y = 9$ and $h = 41$, determine x.
   SOLUTION: 40
4. If $y = 9$ and $h = 41$, determine $\cos\theta$.
   SOLUTION: 40/41
5. If $y = 9$ and $h = 41$, determine $\cot\theta$.
   SOLUTION: 40/9
6. If $x = 7$ and $y = 24$, determine h.
   SOLUTION: 25
7. If $x = 7$ and $y = 24$, determine $\tan\theta$.
   SOLUTION: 24/7
8. If $x = 7$ and $y = 24$, determine $\sin\theta$.
   SOLUTION: 24/25
9. If $x = 7$ and $y = 24$, determine $\sin^2\theta$.
   SOLUTION: $(24/25)^2$
10. If $x = 7$ and $y = 24$, determine $\sin^2\theta + \cos^2\theta$.
    SOLUTION: 1
11. If $\sin\theta = 3/5$ and $x = 4$, determine h.
    SOLUTION: 5
12. If $\cos\theta = 3/5$ and $h = 5$, determine x.
    SOLUTION: 4
13. If $\theta = \pi/4$ and $h = 1$, determine x.
    SOLUTION: $\sqrt{2}/2$
14. If $\theta = \pi/4$and $h = 2$, determine x.
    SOLUTION: $\sqrt{2}$
15. If $\theta = \pi/3$ and $h = 2$, determine y.
    SOLUTION: $\sqrt{3}$

Simplify

1. $\frac{2}{7}+\frac{3}{9}$
   SOLUTION: $\frac{13}{21}$

2. $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$
   SOLUTION: $\frac{bc+ac+ab}{abc}$

3. $(a+b)^2$
   SOLUTION: $a^2+2ab+b^2$

4. $\frac{a}{b}+\frac{c}{d}$
   SOLUTION: $\frac{ad+bc}{bd}$

5. $\frac{a^2-ab}{b-a}$
   SOLUTION: $-a$ if $a \neq b$

6. $\frac{1/a-2/b}{1/a}$
   SOLUTION: $1-\frac{2a}{b}$

7. $(a+b)(a-b)$
   SOLUTION: a^2-b^2

8. $\frac{a}{b}-\frac{b}{a}$
   SOLUTION: $\frac{a^2-b^2}{ab}$

9. $\frac{a+b}{5}-\frac{1}{1-a}$
   SOLUTION: $\frac{a-a^2+b-ab+5}{5-5a}$

10. $\frac{x^2-(x+a)^2}{a}$
    SOLUTION: $-2x-a$

11. $\frac{x^{10}x^{10a}}{x^b x^{2a}}$
    SOLUTION: $x^{10+8a-b}$

12. $\frac{\log_2(6)}{\log_2(3)}$
    SOLUTION: Cannot be simplified

13. $\log_2(16)-\log_3(27)$
    SOLUTION: 1

14. $\log_a(a^{100})-\log_2(2^{95})$
    SOLUTION: 5

Find the equation of the line through the given points

1. $P_1(0,0), P_2(a,b)$
   SOLUTION: $(y-b)=\frac{b}{a}(x-a)$, if $a \neq 0$

2. $P_1(1,0), P_2(2,0)$
   SOLUTION: $y=0$

3. $P_1(0,1), P_2(0,2)$
   SOLUTION: $x = 0$

4. $P_1(a,c), P_2(c,a)$ (assume $a \neq c$)
   SOLUTION: $(y - c) = -1(x - a)$

Find the roots of the following, using the quadratic equation if you can't do it by hand

1. $6x^2 - 36x + 35$
   SOLUTION: $x = \dfrac{36 \pm \sqrt{456}}{12}$

2. $2x^2 - x - 1$
   SOLUTION: 1 and -1/2

3. $4x^2 + 4x + 1$
   SOLUTION: -1/2

4. $x^2 + x + 1$
   SOLUTION: There are no real roots.

Answer the following trigonometry questions

1. Suppose that $\sin\theta = -\frac{5}{12}$ and $\frac{3\pi}{2} < \theta < 2\pi$. What is $\cos\theta$?
   SOLUTION: $\sqrt{119}/12$

2. Suppose that $\sin\theta = -\frac{5}{12}$ and $\pi < \theta < \frac{3\pi}{2}$. What is $\cos\theta$?
   SOLUTION: $-\sqrt{119}/12$

3. Suppose that $\cos\theta = \frac{1}{7}$ and $\frac{3\pi}{2} < \theta < 2\pi$. What is $\sin\theta$?
   SOLUTION: $-\sqrt{48}/7$

4. Suppose that $\cos\theta = -\frac{1}{7}$ and $\pi < \theta < \frac{3\pi}{2}$. What is $\sin\theta$?
   SOLUTION: $-\sqrt{48}/7$