

Music Is Mathematical, Mathematics Is Musical

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What To Expect

Out of tune? Blame Pythagoras and Euclid.

The chromatic scale and doughnuts

Odd harmonics and jazz

Why are mathematicians often musicians, but not vice-versa?

Pythagorean harmonious intervals: The good news

Interval	Frequency Ratio	Example
The Octave	$2/1$	C:c
The Fifth	$3/2$	C:G
The Fourth	$4/3$	C:F
The Major Third	$5/4$	C:E
The Minor Third	$6/5$	C:E \flat
The Major Tone	$9/8$	C:D
The Lesser Tone	$10/9$	D:E
The Diatonic Semitone	$16/15$	C:D \flat
The Chromatic Semitone	$25/24$	C:C \sharp

The good news: All of these can be played on a violin.

What's Wrong With This Picture?

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On the piano, those notes are the same.

$$\therefore (3/2)^{12} = 2^7, \text{ so } 3^{12} = 2^{19} \dots$$

The Piano Theorem: If pianos are tuned so that octaves and fifths are perfect, then

$$3^{12} = 2^{19}.$$

Pythagorean harmonious intervals: The bad news

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The bad news: Only the octave can be played on a piano.

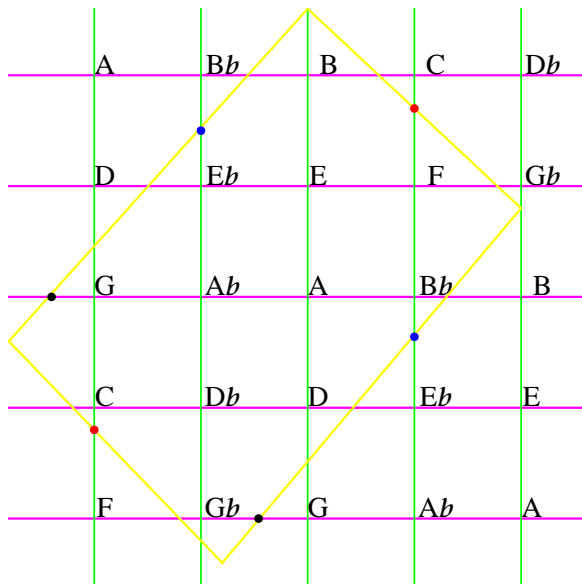
Frequencies of Harmonics of Low A

Harmonic	Note	Natural Pitch	Piano Pitch
1	A2	110	110
2	A3	220	220
3	E4	330	329.6
4	A4	440	440
5	C#5	550	554.4
6	E5	660	659.3
7	G5	770	783.0
8	A5	880	880
9	B5	990	987.8
10	C#6	1100	1109.0
11	D#6	1210	1244.5
12	E6	1320	1318.5

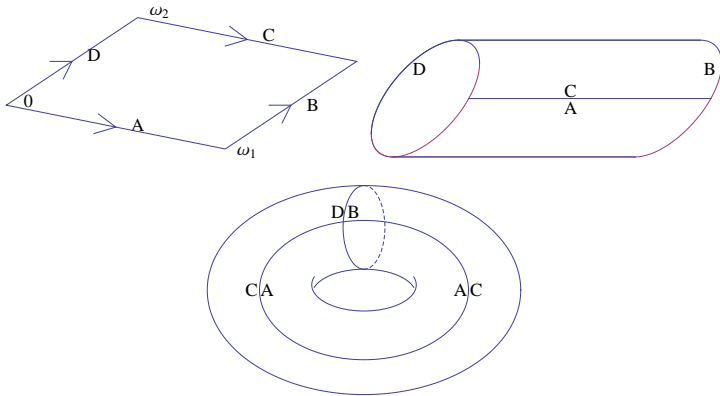
Chromatic scale and circle of fifths in the plane

A	B <i>b</i>	B	C	D <i>b</i>
D	E <i>b</i>	E	F	G <i>b</i>
G	A <i>b</i>	A	B <i>b</i>	B
C	D <i>b</i>	D	E <i>b</i>	E
F	G <i>b</i>	G	A <i>b</i>	A

Chromatic and fifths in the plane: Ready to fold

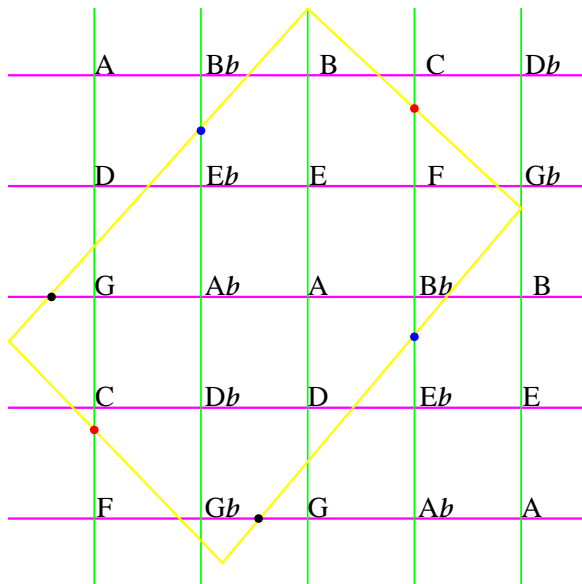


Folding a parallelogram into a doughnut

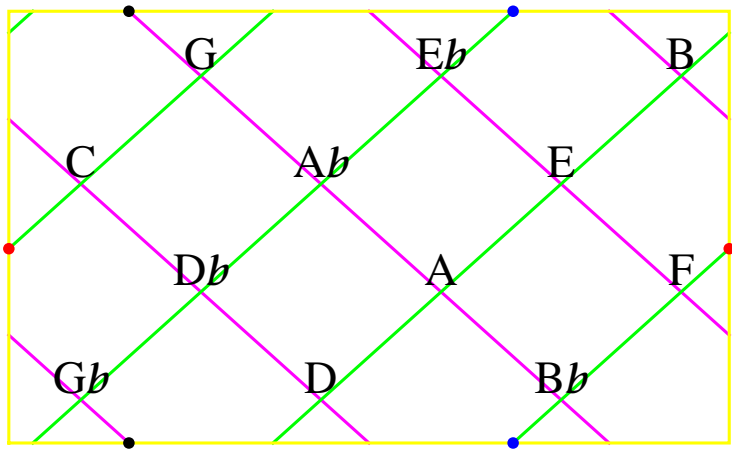


A parallelogram folds into a doughnut

Chromatic and fifths in the plane: Still ready to fold



Chromatic scale and circle of fifths on a doughnut



Chromatic/fifths on a doughnut

Archytas of Tarentum's Quadrivium

	Magnitudes	Numbers
At Rest		
In Motion		

Archytas of Tarentum's Quadrivium

	Magnitudes	Numbers
At Rest	Geometry	
In Motion		

Archytas of Tarentum's Quadrivium

	Magnitudes	Numbers
At Rest	Geometry	Arithmetic
In Motion		

Archytas of Tarentum's Quadrivium

	Magnitudes	Numbers
At Rest	Geometry	Arithmetic
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Archytas of Tarentum's Quadrivium

	Magnitudes	Numbers
At Rest	Geometry	Arithmetic
In Motion	Astronomy	Music

THANK YOU!